

# ON GEOMETRICALLY EQUIVALENT $S$ -ACTS

YEFIM KATSOV

**ABSTRACT.** In this paper, considering the geometric equivalence for algebras of a variety  ${}_S\mathcal{A}$  of  $S$ -acts over a monoid  $S$ , we obtain representation theorems describing all types of the equivalence classes of geometrically equivalent  $S$ -acts of varieties  ${}_S\mathcal{A}$  over groups  $S$ .

## 1. INTRODUCTION

In the last decade, by Prof. B. I. Plotkin and his school, algebraic geometry was introduced in a quite general setting — namely, algebraic geometry over algebras of an arbitrary variety of universal algebras (not only of group varieties as, for example, in [1]) — that brought forth a fascinating new area of algebra known as universal algebraic geometry (see, *e.g.* [11], [12], [2], [14], and [6] for details). And one of the principal problems in algebraic geometry over algebras of a variety  $\Theta$  of universal algebras involves studying interrelations between relations between algebras  $G_1$  and  $G_2$  of  $\Theta$  and relations between  $G_1$ - and  $G_2$ -geometries over them. As has repeatedly been emphasized in [15], [14], [12], and [11], the fundamental notion of *geometric equivalence*, heavily based on congruence theories of finitely generated free algebras of the variety  $\Theta$ , proves to be crucial and important in all investigations concerning this problem (one may consult [13], [3], [4], [17], and [16] for activity, obtained results and open problems in this direction).

In this paper, continuing studying problems associated with algebraic geometry over algebras of varieties  ${}_S\mathcal{A}$  of  $S$ -acts over monoids  $S$ , initiated in [7], we consider the geometric equivalence of algebras in the varieties  ${}_S\mathcal{A}$ . After introducing some notions and facts of algebraic geometry over algebras of  ${}_S\mathcal{A}$  needed in a sequence, in Section 3, among other results, we obtain the main results of the paper — the representation theorems (Theorems 3.21 and 3.22) — describing all types of the equivalence classes of geometrically equivalent  $S$ -acts

---

1991 *Mathematics Subject Classification.* Primary 20M30, 20M99; Secondary 08C05, 20M50.

*Key words and phrases.*  $S$ -acts over monoids, geometric equivalence of algebras, universal algebraic geometry.

of the varieties  ${}_S\mathcal{A}$  over groups  $S$ . We conclude the paper by stating three open problems, delineating quite interesting and promising, in our view, directions for further investigations.

Finally, all notions and facts of categorical algebra, used here without any comments, can be found in [9]; for notions and facts from universal algebra, we refer to [5].

## 2. BASIC PRELIMINARY NOTIONS AND FACTS OF ALGEBRAIC GEOMETRY OVER $S$ -ACTS

**2.1. Algebraic varieties and closed congruences.** Recall (see, *e.g.* [8]) that a *left  $S$ -act* over a monoid  $S$  is a non-empty set  $A$  together with a scalar multiplication  $(s, a) \mapsto sa$  from  $S \times A$  to  $A$  such that  $1_S a = a$  and  $(st)a = s(ta)$  for all  $s, t \in S$  and  $a \in A$ . *Right  $S$ -act* over  $S$  and homomorphisms between  $S$ -acts are defined in the standard manner. And, from now on, let  ${}_S\mathcal{A}$  denote the category of left  $S$ -acts over a monoid  $S$ . Then (see, for example, [8]), a *free (left)  $S$ -act* with a set of free generators, or a basis set,  $I$  is a coproduct  $\coprod_{i \in I} S_i, S_i \cong {}_S S, i \in I$ , of the copies of  ${}_S S$  in the category  ${}_S\mathcal{A}$ . It is obvious that two free  $S$ -acts  $\coprod_{i \in I} S_i$  and  $\coprod_{j \in J} S_j$  are isomorphic in  ${}_S\mathcal{A}$  iff  $|I| = |J|$ , and, hence, by [6, Definition 2.8]  ${}_S\mathcal{A}$  is an IBN-variety. Let  $X_0 = \{x_1, x_2, \dots, x_n, \dots\} \subseteq \mathcal{U}$  be a fixed denumerable set of an infinite universe  $\mathcal{U}$ , and  ${}_S\mathcal{A}^0$  the full subcategory of free  $S$ -acts  $F_X \stackrel{\text{def}}{=} \coprod_{x \in X} S_x, X \subseteq \mathcal{U}, |X| < \infty$ , with a finite basis  $X$  of the variety  ${}_S\mathcal{A}$ .

For a fixed  $S$ -act  $G \in |{}_S\mathcal{A}|$  of the variety  ${}_S\mathcal{A}$  and a free algebra  $F_X \in |{}_S\mathcal{A}^0|$ , the set  ${}_S\mathcal{A}(F_X, G)$  of homomorphisms from  $F_X$  to  $G$  is treated as an affine space over  $G$  consisting of points (homomorphisms)  $\mu : F_X \longrightarrow G$ . Any system  $T = \{s_{x_i} = t_{y_i} \mid x_i, y_i \in X, s_{x_i} \in S_{x_i}, t_{y_i} \in S_{y_i}, i \in I\}$  of the equations in free variables of  $X$  in  $F_X$  can be obviously viewed as the binary relation  $T \subseteq F_X \times F_X$  on  $F_X$ , and a point (homomorphism)  $\mu : F_X \longrightarrow G$  is a *solution* of  $T$  iff  $\mu(s_{x_i}) = \mu(t_{y_i})$  for any equation  $s_{x_i} = t_{y_i}$  of  $T$ , *i.e.*, iff  $T \subseteq \text{Ker } \mu$ . Thus, for any set of points  $A \subseteq {}_S\mathcal{A}(F_X, G)$  and any binary relation  $T \subseteq F_X \times F_X$  on  $F_X$ , the assignments

$$T \longmapsto T'_G \stackrel{\text{def}}{=} \{\mu : F_X \longrightarrow G \mid T \subseteq \text{Ker } \mu\} \text{ and } A \longmapsto A' \stackrel{\text{def}}{=} \cap_{\mu \in A} \text{Ker } \mu$$

define the Galois correspondence between binary relations (or systems of equations)  $T$  on  $F_X$  and sets of points  $A$  of the space  ${}_S\mathcal{A}(F_X, G)$ . A congruence  $T$  on  $F_X$  is said to be  *$G$ -closed* if  $T = A'$  for some point set  $A \subseteq {}_S\mathcal{A}(F_X, G)$ ; a point set  $A$  is called a  *$G$ -closed set*, or an *algebraic variety*, in the space  ${}_S\mathcal{A}(F_X, G)$  if  $A = T'_G$  for some relation  $T$  on  $F_X$ . As usual, the Galois correspondence produces the closures:

$A'' \stackrel{\text{def}}{=} (A')'$  and  $T'' \stackrel{\text{def}}{=} (T'_G)'$ . And one easily obtains the following observation.

**Proposition 2.1.** *A congruence  $T \subseteq F_X \times F_X$  on  $F_X$  is  $G$ -closed iff  $T = T''$ .*

**Proof.**  $\implies$ . Let  $T = A'$  for a point set  $A \subseteq {}_s\mathcal{A}(F_X, G)$ , i.e.,  $T = \cap_{\mu \in A} \text{Ker} \mu$ . Then,  $T'_G = \{\lambda : F_X \longrightarrow G \mid \cap_{\mu \in A} \text{Ker} \mu = T \subseteq \text{Ker} \lambda\}$ , and hence,  $T'' = \cap_{\lambda \in T'_G} \text{Ker} \lambda = \cap_{\mu \in A} \text{Ker} \mu \cap_{\lambda \in T'_G \setminus A} \text{Ker} \lambda = \cap_{\mu \in A} \text{Ker} \mu = T$ .

$\impliedby$ .  $T = T'' = (T'_G)' = A'$ , where  $A = T'_G$ .  $\square$

**Corollary 2.2.** (cf. [14, Proposition 2.1]) *A congruence  $T \subseteq F_X \times F_X$  on  $F_X$  is  $G$ -closed iff there is an embedding  $\mu : F_X/T \hookrightarrow G^I$  for some set  $I$ .*

**Proof.**  $\implies$ . Since  $T = T'' = (T'_G)'$ , by [5, Theorem 20.2]  $F_X/T$  is a subdirect product of the  $S$ -subacts  $F_X/\text{Ker} \lambda \subseteq G$  of  ${}_sG$ ,  $\lambda \in T'_G$ , and therefore, there exists an embedding  $F_X/T \hookrightarrow G^I$  for the set  $I = T'_G$ .

$\impliedby$ . If  $\mu : F_X/T \hookrightarrow G^I$ , and  $\pi_i : G^I \twoheadrightarrow G$ ,  $i \in I$ , are the canonical projections, then  $T = \cap_{i \in I} \text{Ker} \pi_i \mu$ .  $\square$

Analogously, we have

**Proposition 2.3.** *A point set  $A \subseteq {}_s\mathcal{A}(F_X, G)$  is an algebraic variety iff  $A = A''$ .*

**Proof.**  $\implies$ . Let  $A = T'_G = \{\mu : F_X \longrightarrow G \mid T \subseteq \text{Ker} \mu\}$  for some relation  $T \subseteq F_X \times F_X$  on  $F_X$ . Then,  $A' = \cap_{\mu \in A} \text{Ker} \mu \supseteq T$  and, hence,  $A'' = \{\lambda : F_X \longrightarrow G \mid T \subseteq \cap_{\mu \in A} \text{Ker} \mu \subseteq \text{Ker} \lambda\}$ , and therefore,  $A'' \subseteq A \subseteq A''$ .

$\impliedby$ . Obvious, as  $A = A'' = (A' = T)'$ .  $\square$

**2.2. The functors  $\text{Alv}_G$  and  $\text{Cl}_G$  over  ${}_s\mathcal{A}^0$ .** We consider two natural functors:  $\text{Alv}_G : {}_s\mathcal{A}^0 \longrightarrow \text{Set}$  and  $\text{Cl}_G : ({}_s\mathcal{A}^0)^{\text{op}} \longrightarrow \text{Set}$ .

a)  $\text{Alv}_G : {}_s\mathcal{A}^0 \longrightarrow \text{Set}$ . For any  $F_X \in |{}_s\mathcal{A}^0|$ ,  $\text{Alv}_G(F_X) \stackrel{\text{def}}{=} \{A \mid A \subseteq {}_s\mathcal{A}(F_X, G) \text{ \& } A = A''\}$ ; and for any homomorphism  $s \in {}_s\mathcal{A}^0(F_Y, F_X)$  and  $B = \{\mu : F_Y \longrightarrow G \mid B' \subseteq \text{Ker} \mu\} \in \text{Alv}_G(F_Y)$ ,  $\text{Alv}_G(s) \stackrel{\text{def}}{=} s^{-1}B \stackrel{\text{def}}{=} \{\alpha : F_X \longrightarrow G \mid \alpha s \in B\} = \{\alpha : F_X \longrightarrow G \mid sB' \subseteq \text{Ker} \alpha\}$ , where the relation  $sB' \stackrel{\text{def}}{=} \{(\omega, \omega') \in F_X \times F_X \mid \exists (\omega_0, \omega'_0) \in B' \subseteq F_Y \times F_Y : s(\omega_0) = \omega \text{ \& } s(\omega'_0) = \omega'\}$ .

b)  $\text{Cl}_G : ({}_s\mathcal{A}^0)^{\text{op}} \longrightarrow \text{Set}$ . For any  $F_X \in |{}_s\mathcal{A}^0|$ ,  $\text{Cl}_G(F_X) \stackrel{\text{def}}{=} \{T \mid T \subseteq F_X \times F_X \text{ \& } T = T''\}$ ; and for any homomorphism  $s \in {}_s\mathcal{A}^0(F_Y, F_X)$

and  $T = T'' = (T'_G)' = \cap_{\alpha \in T'_G} \text{Ker } \alpha \subseteq F_X \times F_X$ ,  $Cl_G(s) \stackrel{\text{def}}{=} s^{-1}T \stackrel{\text{def}}{=} \cap_{\alpha \in T'_G} \text{Ker } s\alpha \subseteq F_Y \times F_Y$ .

One can easily see that the just defined mappings  $Alv_G$  and  $Cl_G$  are indeed covariant and contravariant functors, respectively. Also, from the results of [14, Section 2.2] it is easy to see that for any  $F_X \in |_S\mathcal{A}^0|$  the sets  $Alv_G(F_X)$  and  $Cl_G(F_X)$  are meet-semilattices with the respect to the natural partial orders on the set of all subsets of  $_S\mathcal{A}(F_X, G)$  and  $F_X \times F_X$ , respectively. Moreover, if we define  $A \overline{U} B \stackrel{\text{def}}{=} (A \cup B)''$  and  $T_1 \overline{U} T_2 \stackrel{\text{def}}{=} (T_1 \cup T_2)''$  for any  $A, B \in Alv_G(F_X)$  and  $T_1, T_2 \in Cl_G(F_X)$ , the sets  $Alv_G(F_X)$  and  $Cl_G(F_X)$  become lattices, and by [14, Proposition 2.4] (see also [12, Proposition 4]) the assignments  $T \mapsto T'$  for each  $T \in Cl_G(F_X)$  establish a dual isomorphism between the lattices  $Cl_G(F_X)$  and  $Alv_G(F_X)$ . Following [14, Definition 2.4], we call an  $S$ -act  $G$  *geometrically stable* iff  $A \overline{U} B = A \cup B$  for any  $F_X \in |_S\mathcal{A}^0|$  and  $A, B \in Alv_G(F_X)$ , or, as one can easily see by using the results of [14, Section 2.2], iff  $(T'_1 \cup T'_2) = (T_1 \cap T_2)'$  for any  $F_X \in |_S\mathcal{A}^0|$  and  $T_1, T_2 \in Cl_G(F_X)$ . Thus, considering a particular case of the varieties  $_S\mathcal{A}$ , namely the case when the monoid  $S = \{1\}$ , and therefore,  $_S\mathcal{A}$  is just the category *Set* of non-empty sets, and in contrast to [12, Theorem 1], we have the following observation.

**Proposition 2.4.** *In Set, the singletons are the only geometrically stable sets.*

**Proof.** Let  $G, X \in \text{Set}$ , and  $|G| = 1$ . As it is clear that all congruences on  $X$  are just equivalence relations,  $Cl_G(X)$  has only the universal equivalence relation, and  $|Alv_G(X)| = 1$ ,  $G$  is geometrically stable.

Let  $G, X \in \text{Set}$ ,  $|G| \geq 2$ , and  $T$  an equivalence relation on  $X$ . Then from obvious and well-known set theory facts, there are embeddings  $X/T \hookrightarrow 2^{|X/T|} \hookrightarrow G^{|X/T|}$ , and therefore, by Corollary 2.2 the equivalence relation  $T$  is  $G$ -closed. Now take  $X = \{x_1, x_2, x_3\}$ , and let  $T_1 = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_1, x_2), (x_2, x_1)\}$  and  $T_2 = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_2, x_3), (x_3, x_2)\}$  be equivalence relations on  $X$ . Then,  $(T_1 \cap T_2) = \{(x_1, x_1), (x_2, x_2), (x_3, x_3)\}$  and  $(T_1 \cap T_2)' = \text{Set}(X, G)$ ; and  $T'_1 = \{\alpha : X \rightarrow G \mid \alpha(x_1) = \alpha(x_2)\}$ ,  $T'_2 = \{\beta : X \rightarrow G \mid \beta(x_2) = \beta(x_3)\}$ . For  $g_1, g_2 \in G$ ,  $g_1 \neq g_2$ , there exists  $\gamma : X \rightarrow G$  such that  $\gamma(x_1) = \gamma(x_3) = g_1$  and  $\gamma(x_2) = g_2$ . Thus,  $\gamma \in (T_1 \cap T_2)'$ , but  $\gamma \notin (T'_1 \cup T'_2)$ .

□

### 3. GEOMETRICALLY EQUIVALENT $S$ -ACTS

For varieties of  $S$ -acts, the notion of *geometric equivalence* is defined as follows.

**Definition 3.1.** (cf. [14, Definition 3.1] and [12]) Two  $S$ -acts  $G_1, G_2 \in |\mathcal{S}\mathcal{A}|$  are said to be *geometrically equivalent*,  $G_1 \overset{\Delta}{\sim} G_2$ , iff  $T''_{G_1} = T''_{G_2}$  for any binary relation  $T \subseteq F_X \times F_X$  on any free  $S$ -acts  $F_X \in |\mathcal{S}\mathcal{A}^0|$ , i.e., iff  $Cl_{G_1}(F_X) = Cl_{G_2}(F_X)$  for any free  $S$ -acts  $F_X \in |\mathcal{S}\mathcal{A}^0|$ , or iff  $Cl_{G_1} = Cl_{G_2}$  as functors.

From the proof of Proposition 2.4 one immediately obtains the following description of geometrically equivalent classes in *Set*.

**Theorem 3.2.** *In Set, there are only two classes of geometrically equivalent sets: singletons, and all sets with more than one element.*

□

**Lemma 3.3.** *For any relation  $T \subseteq F_X \times F_X$  and monomorphism  $\mu : G_1 \rightarrow G_2$ ,  $T''_{G_1} \supseteq T''_{G_2}$ .*

**Proof.** Indeed,  $T'_{G_1} = \{\alpha : F_X \rightarrow G_1 \mid T \subseteq \text{Ker } \alpha = \text{Ker } \mu\alpha\}$ ,  $T'_{G_2} = \{\beta : F_X \rightarrow G_2 \mid T \subseteq \text{Ker } \beta\} \supseteq \{\mu\alpha : F_X \rightarrow G_2 \mid \alpha : F_X \rightarrow G_1 \text{ \& } T \subseteq \text{Ker } \alpha\}$ . Therefore,  $T''_{G_1} = \bigcap_{\alpha \in T'_{G_1}} \text{Ker } \alpha = \bigcap_{\alpha \in T'_{G_1}} \text{Ker } \mu\alpha \supseteq \bigcap_{\beta \in T'_{G_2}} \text{Ker } \beta = T''_{G_2}$ . □

**Corollary 3.4.** ([14, Proposition 3.1])  $G \overset{\Delta}{\sim} G^I$  for any  $G$  and any set  $I$ .

**Proof.** Because of the diagonal embedding  $G \rightarrow G^I$ , it follows from Lemma 3.3 that  $T''_G \supseteq T''_{G^I}$  for any relation  $T \subseteq F_X \times F_X$ .

Now let  $\pi_i : G^I \rightarrow G_i$ ,  $G_i = G$ ,  $i \in I$ , be the natural projections,  $T$  a binary relation on  $F_X$ , and  $\alpha \in T'_{G^I}$ . Then,  $T \subseteq \text{Ker } \pi_i \alpha$  for any  $i \in I$ , and, hence,  $\pi_i \alpha \in T'_G$ ,  $i \in I$ . And it is clear that  $\text{Ker } \alpha = \bigcap_{i \in I} \text{Ker } \pi_i \alpha \supseteq \bigcap_{\beta \in T'_G} \text{Ker } \beta = T''_G$ , and therefore,  $T''_{G^I} = \bigcap_{\alpha \in T'_{G^I}} \text{Ker } \alpha \supseteq T''_G$ . □

The next observation will prove to be a quite useful generalization of Corollary 3.4.

**Corollary 3.5.** *Let for  $G_1$  and  $G_2$  there exist sets  $I$  and  $J$  and embeddings  $G_1 \rightarrow G_2^I$  and  $G_2 \rightarrow G_1^J$ . Then,  $G_1 \overset{\Delta}{\sim} G_2$ .* □

Now let  $\mathcal{S}\mathcal{B} \subseteq \mathcal{S}\mathcal{A}$  be a subvariety of the variety  $\mathcal{S}\mathcal{A}$ . As is well-known (see, for example, [10, Chapter VI]), a free algebra with a finite basis  $X$  in the variety  $\mathcal{S}\mathcal{B}$  is, in fact, the factor algebra  $F_X/\tau$  of  $F_X \in |\mathcal{S}\mathcal{A}^0|$

modulo  $\tau$ , where  $\tau$  is a corresponding verbal congruence. For a congruence  $T$  on  $F_X/\tau$  and the epimorphism  $\pi : F_X \twoheadrightarrow F_X/\tau$  corresponding  $\tau$ , let  $\pi^{-1}T \stackrel{\text{def}}{=} \{(\omega, \omega') \in F_X \times F_X \mid (\pi(\omega), \pi(\omega')) \in T\}$  be the inverse image of the congruence  $T$ . In these notations, we have the following observation.

**Proposition 3.6.** *For any  $G \in |_s\mathcal{B}|$ , the mapping  $T \mapsto \pi^{-1}T$  is a bijection between the sets of  $G$ -closed congruences in  $_s\mathcal{B}$  and in  $_s\mathcal{A}$ .*

**Proof.** Let  $T$  be  $G$ -closed in  $_s\mathcal{B}$ . Then,  $T'_G = \{\alpha : F_X/\tau \rightarrow G \mid T \subseteq \text{Ker } \alpha\}$  and  $T = T''_G = \cap_{\alpha \in T'_G} \text{Ker } \alpha$ . One may easily see that the latter is a limit of a corresponding diagram, and therefore, applying the results of [9, Section 9.8] on iterated limits, obtain that  $\pi^{-1}T = \cap_{\alpha \in T'_G} \text{Ker } \alpha\pi$ , and, hence,  $\pi^{-1}T$  is  $G$ -closed in  $_s\mathcal{A}$ , too.

Now, if a congruence  $T$  on  $F_X$  is  $G$ -closed, then, since  $G \in |_s\mathcal{B}|$ , one has  $T = \cap_{\alpha \in A} \text{Ker } \alpha\pi$  for some  $A \subseteq {}_s\mathcal{B}(F_X/\tau, G)$ , and, again taking into consideration the iterated limits, obtains that  $T = \pi^{-1}(\cap_{\alpha \in A} \text{Ker } \alpha)$ . As  $\cap_{\alpha \in A} \text{Ker } \alpha$  is a  $G$ -closed congruence on  $F_X/\tau$ , we conclude that any  $G$ -closed congruence on  $F_X$  is an inverse image of  $G$ -closed congruence on  $F_X/\tau$ .

Noting that  $\pi(\pi^{-1}T) = T$  for any congruence  $T$  on  $F_X/\tau$ , we end the proof.  $\square$

**Corollary 3.7.** ([14, Proposition 3.2]) *For any two algebras  $G_1, G_2 \in |_s\mathcal{B}|$  of a subvariety  $_s\mathcal{B}$  of the variety  $_s\mathcal{A}$ ,  $G_1 \stackrel{\Delta}{\sim} G_2$  in  $_s\mathcal{A}$  iff  $G_1 \stackrel{\Delta}{\sim} G_2$  in  $_s\mathcal{B}$ .*  $\square$

Defining an  $S$ -act  $A$  to be *trivial* iff  $sa = a$  for any  $s \in S$  and  $a \in A$ , one readily sees that all trivial  $S$ -acts form a subvariety of the variety  $_s\mathcal{A}$ , defined by the identities  $\forall x(sx = x)$ ,  $s \in S$ , and actually coinciding with  $\text{Set}$ . From this observation, Corollary 3.7, and Theorem 3.2 we have

**Corollary 3.8.** *In  $_s\mathcal{A}$ , all trivial  $S$ -acts of cardinality greater than one are geometrically equivalent.*  $\square$

As usual, an  $S$ -act  $A \in |_s\mathcal{A}|$  is *cyclic* if there is  $a \in A$  such that  ${}_sA = \{Sa\}$ , and  ${}_sA$  is called *simple* if it has no proper subacts. It is obvious that any simple act is cyclic.

From now on, we assume that a monoid  $S$  is a group. Then, it is clear that any cyclic act is simple, and, by [8, Proposition 1.5.34], any act  $A \in |_s\mathcal{A}|$  is a disjoint union, or a coproduct in the category  $_s\mathcal{A}$ , of simple (cyclic) subacts. Moreover, using [8, Proposition 1.5.17], one can easily see that for any cyclic act  ${}_s\overline{G}$  there exists a subgroup  $G$  of  $S$

such that  ${}_S\overline{G}$  is isomorphic to the act  ${}_SS/G$  of the left cosets modulo  $G$ ; and, therefore, any  $A \in |{}_S\mathcal{A}|$ , in fact, can be considered as a coproduct of suitable left cosets  ${}_SS/G \stackrel{\text{def}}{=} {}_S\overline{G}$  of the group  $S$ .

**Proposition 3.9.** *Let  $G$  and  $H$  be subgroups of a group  $S$ , and the cyclic  $S$ -acts  ${}_SS/G = {}_S\overline{G}$  and  ${}_SS/H = {}_S\overline{H}$  geometrically equivalent. Then there are  $\alpha, \beta \in S$  such that  $G^\alpha \stackrel{\text{def}}{=} \alpha^{-1}G\alpha \subseteq H$  and  $H^\beta \stackrel{\text{def}}{=} \beta^{-1}H\beta \subseteq G$ .*

**Proof.** Let  $T_1 = \{(sg_1, sg_2) \mid g_1, g_2 \in G, s \in S\}$ ,  $T_2 = \{(sh_1, sh_2) \mid h_1, h_2 \in H, s \in S\}$  be the congruences on the act  ${}_SS$ , and  $\pi_1 : {}_SS \longrightarrow {}_SS/T_1 = {}_SS/G = {}_S\overline{G}$ ,  $\pi_2 : {}_SS \longrightarrow {}_SS/T_2 = {}_SS/H = {}_S\overline{H}$  the canonical epimorphisms corresponding to the subgroups  $G, H \subseteq S$ , respectively.

As  $T_1 = \text{Ker } \pi_1$ ,  $T_1 \in \text{Cl}_{\overline{G}}({}_SS)$ , and, since  $\overline{G} \stackrel{\Delta}{\sim} \overline{H}$ ,  $T_1 \in \text{Cl}_{\overline{H}}({}_SS)$ , too. Therefore, there exists a homomorphism  $f : {}_SS \longrightarrow {}_S\overline{H}$  such that  $\text{Ker } f \supseteq T_1$ , and, hence, there is such a homomorphism  $\overline{\alpha} : {}_S\overline{G} \longrightarrow {}_S\overline{H}$  that  $f = \overline{\alpha}\pi_1$ . From this and noting that the free object  ${}_SS \in |{}_S\mathcal{A}|$  is obviously projective in the category  ${}_S\mathcal{A}$  (see, also [8, Proposition III.17.2]), we have  $\overline{\alpha}\pi_1 = \pi_2\alpha$  for some  $\alpha : {}_SS \longrightarrow {}_SS$ . Agreeing to write endomorphisms of  ${}_SS$  on the right of the elements they act on, one can easily see that actions of endomorphisms actually coincide with multiplications of elements of  ${}_SS$  on the right by elements of the monoid  $S$ , and, therefore,  $\text{End}({}_SS) = S$ , and, thus,  $\forall s \in S : \alpha(s) \stackrel{\text{def}}{=} s\alpha$ , where  $\alpha \in S$ . Then, from the equation  $\overline{\alpha}\pi_1 = \pi_2\alpha$ , we have:  $(sg_1, sg_2) \in T_1 \implies (sg_1\alpha, sg_2\alpha) \in T_2 \iff g_1\alpha h = g_2\alpha$  for some  $h \in H \iff (\alpha^{-1}g_1\alpha)h = \alpha^{-1}g_2\alpha$  for some  $h \in H$ . Therefore, we have that  $G^\alpha = \alpha^{-1}G\alpha \subseteq tH$  for some  $t \in S$ . Show that, in fact,  $t \in H$ . Indeed, if  $g_1, g_2 \in G$ , then there exist  $h_1, h_2, h_3 \in H$  such that  $\alpha^{-1}g_1\alpha = th_1, \alpha^{-1}g_2\alpha = th_2, \alpha^{-1}g_1g_2\alpha = th_3$ , and, hence,  $th_3 = th_1th_2$  from which immediately follows that  $t \in H$ . Thus, we have proved an inclusion  $G^\alpha = \alpha^{-1}G\alpha \subseteq H$  for some  $\alpha \in S$ .

By symmetry, we also obtain an inclusion  $H^\beta = \beta^{-1}H\beta \subseteq G$  for some  $\beta \in S$ .  $\square$

**Proposition 3.10.** *Let  $G$  and  $H$  be subgroups of a group  $S$ , and  $H \subset G$ . Then,  ${}_S\overline{H} \stackrel{\Delta}{\approx} {}_S\overline{G}$ .*

**Proof.** Let  $\pi_{\overline{G}} : {}_SS \longrightarrow {}_S\overline{G}$ ,  $\pi_{\overline{H}} : {}_SS \longrightarrow {}_S\overline{H}$  be the canonical epimorphisms corresponding to the subgroups  $G, H \subseteq S$ , respectively. Then,  $T = \text{Ker } \pi_{\overline{H}} = \{(s, t) \mid s, t \in S \text{ \& } sH = tH\}$  obviously is an  ${}_S\overline{H}$ -closed congruence, i.e.,  $T = T''_{\overline{H}}$ .

Hence, assuming  ${}_S\overline{H} \triangleq {}_S\overline{G}$ , we have that  $T$  is an  ${}_S\overline{G}$ -closed congruence, too; and therefore, there exist a family of homomorphisms  $f_i : {}_SS \rightarrow {}_S\overline{G}, i \in I$ , such that  $T = \cap_{i \in I} \text{Ker } f_i$ . However, using the Yoneda lemma (see, for example, [9, Section III.2]) and the fact that  ${}_SS$  is a projective(free) object in the category  ${}_S\mathcal{A}$ , one immediately has that for any  $i \in I$ , there exists an isomorphism  $\alpha_i \in \text{Iso}_{{}_S\mathcal{A}}({}_SS, {}_SS) = S$  such that  $f_i = \pi_{\overline{G}} \alpha_i$ , and therefore,  $\text{Ker } f_i = \text{Ker } \pi_{\overline{H}} \neq T = \text{Ker } \pi_{\overline{H}}$ . Thus,  $T \neq \cap_{i \in I} \text{Ker } f_i$ .  $\square$

**Lemma 3.11.** *Let  $G$  and  $H$  be subgroups of a group  $S$ , and  $G^\alpha = \alpha^{-1}G\alpha = H$  for some  $\alpha \in S$ . Then,  ${}_S\overline{H} \cong {}_S\overline{G}$  in  ${}_S\mathcal{A}$ , and therefore,  ${}_S\overline{H} \triangleq {}_S\overline{G}$ .*

**Proof.** Using the equation  $G\alpha = \alpha H$ , it is enough only to show that assigning  ${}_S\overline{G} \ni sG \mapsto s\alpha H \in {}_S\overline{H}$  for any left  $G$ -coset  $sG \in {}_S\overline{G}$ , one establishes an isomorphism in  ${}_S\mathcal{A}$ . And this fact can easily be checked in a habitual fashion.  $\square$

**Theorem 3.12.** *For any subgroups  $G$  and  $H$  of a group  $S$ ,  ${}_S\overline{H} \triangleq {}_S\overline{G}$  iff  $G^\alpha = H$  for some  $\alpha \in S$ .*

**Proof.**  $\Leftarrow$ . This follows from Lemma 3.11.

$\Rightarrow$ . By Propositions 3.9, 3.10 and using Lemma 3.11, one obtains that  $G^\alpha = H$ .  $\square$

**Corollary 3.13.** *For normal subgroups  $G, H \subseteq S$ ,  $\overline{G} \triangleq \overline{H}$  iff  $G = H$ . In particular, for any subgroups  $G, H$  of an abelian group  $S$ ,  $\overline{G} \triangleq \overline{H}$  iff  $G = H$ .  $\square$*

In what follows, let  $z$ , perhaps with indexes, denote a singleton considered as a trivial  $S$ -act, so to speak, a *zero*  $S$ -act. Using this agreement, we obtain

**Proposition 3.14.** *For any  $S$ -act  $A \in |{}_S\mathcal{A}|$  without zero subacts, the following statements are true:*

- (i)  $A \amalg z_1 \triangleq A \amalg (z_1 \amalg z_2)$ ;
- (ii)  $A \amalg (z_1 \amalg z_2) \triangleq A \amalg (\amalg_{i \in I} z_i)$ , where  $|I| > 1$ .

**Proof.** (i). Let  $F_X = S_{x_1} \amalg S_{x_2}$  for  $X = \{x_1, x_2\}$ , and  $\alpha : F_X \rightarrow A \amalg (z_1 \amalg z_2)$  be defined as  $\alpha(S_{x_1}) = z_1$  and  $\alpha(S_{x_2}) = z_2$ . Then  $\text{Ker } \alpha = \{(s_{x_1}, t_{x_1}) \mid s_{x_1}, t_{x_1} \in S_{x_1}\} \cup \{(u_{x_2}, v_{x_2}) \mid u_{x_2}, v_{x_2} \in S_{x_2}\} \in \text{Cl}_{A \amalg (z_1 \amalg z_2)}(F_X)$ , but it is clear that  $\text{Ker } \alpha \notin \text{Cl}_{A \amalg z_1}(F_X)$ .

(ii). Since  $|I| > 1$ , there is an obvious embedding  $A \amalg (z_1 \amalg z_2) \hookrightarrow A \amalg (\amalg_{i \in I} z_i)$ . Consider  $S$ -act  $[A \amalg (z_1 \amalg z_2)]^I$  and the embedding



$A \coprod (\coprod_{i \in I} z_i) \mapsto [A \coprod (z_1 \coprod z_2)]^I$  defined as  $a \mapsto (a, a, \dots, a, \dots)$  for any  $a \in A$ , and let  $z_i$  go to the string having  $z_1$  in the  $i$ -th place and  $z_2$  everywhere else for any  $i \in I$ . Then, applying Corollary 3.5, we end the proof.  $\square$

Introducing  $A^{(*)I} \stackrel{\text{def}}{=} \coprod_{i \in I} A_i$ ,  $A_i = {}_S A$ ,  $i \in I$ , for any  ${}_S A \in |{}_S \mathcal{A}|$  and any set  $I$ , we also have

**Proposition 3.15.** *For any  $S$ -act  $A \in |{}_S \mathcal{A}|$  without zero subacts,  $B \in |{}_S \mathcal{A}|$ , and  $I$  with  $|I| \geq 1$ , the following statements are true:*

- (i)  $B \coprod A^{(*)I} \coprod z_1 \stackrel{\Delta}{\sim} B \coprod A \coprod z_1$ ;
- (ii)  $B \coprod A^{(*)I} \coprod (z_1 \coprod z_2) \stackrel{\Delta}{\sim} B \coprod A \coprod (z_1 \coprod z_2)$ .

**Proof.** (i). Consider  $S$ -act  $[B \coprod A \coprod z_1]^I$  and the embedding  $B \coprod A^{(*)I} \coprod z_1 \mapsto [B \coprod A \coprod z_1]^I$  such that  $a \in A_i = A \subseteq A^{(*)I}$  goes to the string having  $a$  in the  $i$ -th place and  $z_1$  everywhere else for any  $i \in I$ , and  $z_1 \mapsto (z_1, z_1, \dots, z_1, \dots)$ ,  $b \mapsto (b, b, \dots, b, \dots)$  for any  $b \in B$ . From this, taking into consideration an obvious embedding  $B \coprod A \coprod z_1 \mapsto B \coprod A^{(*)I} \coprod z_1$  and applying Corollary 3.5, we get the statement.

(ii). Just using the embedding  $B \coprod A^{(*)I} \coprod (z_1 \coprod z_2) \mapsto [B \coprod A \coprod (z_1 \coprod z_2)]^I$  such that  $a \in A_i = A \subseteq A^{(*)I}$  goes to the string having  $a$  in the  $i$ -th place and  $z_1$  everywhere else for any  $i \in I$ , and  $z_1 \mapsto (z_1, z_1, \dots, z_1, \dots)$ ,  $z_2 \mapsto (z_2, z_2, \dots, z_2, \dots)$ , and  $b \mapsto (b, b, \dots, b, \dots)$  for any  $b \in B$ , we end the proof in the similar way as in (i).  $\square$

**Proposition 3.16.**  $B \coprod \overline{N}^{(*)I} \stackrel{\Delta}{\sim} B \coprod \overline{N}$  and  $\overline{N}^{(*)I} \stackrel{\Delta}{\sim} \overline{N}$  for any proper normal subgroup  $N \triangleleft S$  of a group  $S$ ,  $B \in |{}_S \mathcal{A}|$ , and  $I$  with  $|I| \geq 1$ .

**Proof.** Since  $N$  is a normal subgroup, one can readily get the embedding  $B \coprod \overline{N}^{(*)I} \mapsto [B \coprod \overline{N}]^I$  such that  $s\bar{1} \in \overline{N}_i = \overline{N} \subseteq \overline{N}^{(*)I}$  goes to the string having  $s\bar{1}$  in the  $i$ -th place and  $s\bar{t}$  everywhere else for any  $i \in I$ , where  $\bar{1}, \bar{t} \in \overline{N}$  are the cosets modulo  $N$  containing  $1 \in S$  and  $t \in S$ , respectively, and  $\bar{1} \neq \bar{t}$ , and  $b \mapsto (b, b, \dots, b, \dots)$  for any  $b \in B$ . From this, taking into consideration an obvious embedding  $B \coprod \overline{N} \mapsto B \coprod \overline{N}^{(*)I}$  and applying Corollary 3.5, we get the statement.  $\square$

**Proposition 3.17.** *For any  $S$ -acts  $A, B \in |{}_S \mathcal{A}|$  without zero subacts, the following statements are true:*

- (i)  $A \stackrel{\Delta}{\sim} B \coprod z$ ;
- (ii)  $A \stackrel{\Delta}{\sim} B \coprod (z_1 \coprod z_2)$ .

**Proof.** (i). Let  $F_X = S_{x_1} \coprod S_{x_2}$  for  $X = \{x_1, x_2\}$ , and  $\alpha : F_X \rightarrow B \coprod z$  be such a homomorphism that  $\alpha(S_{x_1}) \subseteq B$  and  $\alpha(S_{x_2}) = z$ . It is obvious that a non-universal congruence  $\text{Ker } \alpha \in \text{Cl}_{B \coprod z}(F_X)$ ; however,  $\text{Ker } \alpha \notin \text{Cl}_A(F_X)$ .

(ii). The same arguments as in (i) work well for this case, too.  $\square$

**Proposition 3.18.** *For any  $S$ -act  $A \in |_S\mathcal{A}|$  and  $I$  with  $|I| \geq 2$ ,  $A^{(*)I} \overset{\Delta}{\sim} A^{(*)2}$ , where  $A^{(*)2} = A_1 \coprod A_2$  with  $A_1 = A = A_2$ .*

**Proof.** Consider the embedding  $A^{(*)I} \hookrightarrow (A^{(*)2})^I = (A_1 \coprod A_2)^I$  such that  $a \in A_i = A \subseteq A^{(*)I}$  goes to the string having  $a \in A_1$  in the  $i$ -th place and  $a \in A_2$  everywhere else for any  $i \in I$ . Then, using the obvious embedding  $A^{(*)2} \hookrightarrow A^{(*)I}$  and applying Corollary 3.5, we get the statement.  $\square$

**Proposition 3.19.** *Let for  $S$ -acts  $A, B \in |_S\mathcal{A}|$  there exist embeddings  $A \xrightarrow{i_A} B^I$  and  $B \xrightarrow{i_B} A^J$  for some sets  $I$  and  $J$ . Then,  $C \coprod A \overset{\Delta}{\sim} C \coprod B$  for any  $C \in |_S\mathcal{A}|$ .*

**Proof.** Consider the following embeddings  $C \coprod A \xrightarrow{1_C \coprod i_A} C \coprod B^I \xrightarrow{\Delta \times 1} (C \coprod B)^I$ , where the last embedding defined as follows:  $C \ni c \mapsto (c, c, \dots) \in (C \coprod B)^I$  and  $C \coprod B^I \ni (b_i)_{i \in I} \mapsto (b_i)_{i \in I} \in (C \coprod B)^I$ . Then, considering the similarly defined embeddings  $C \coprod B \xrightarrow{1_C \coprod i_B} C \coprod A^J \xrightarrow{\Delta \times 1} (C \coprod A)^J$  and using Corollary 3.5, we end the proof.  $\square$

**Corollary 3.20.** *For any  $S$ -acts  $A, C \in |_S\mathcal{A}|$  and  $I$  with  $|I| \geq 2$ ,  $C \coprod A^{(*)I} \overset{\Delta}{\sim} C \coprod A^{(*)2}$ .*

**Proof.** This follows from the embeddings defined in the proof of Proposition 3.18 and Proposition 3.19.  $\square$

The relation  $\overset{\Delta}{\sim}$  of *geometric equivalence* on the class  $|_S\mathcal{A}|$  of objects of the variety  $_S\mathcal{A}$  is clearly an equivalence relation. Thus, denoting by  $[A] \stackrel{\text{def}}{=} \{ B \mid B \in |_S\mathcal{A}| \text{ \& } B \overset{\Delta}{\sim} A \}$  the class of all  $S$ -acts geometrically equivalent to an  $S$ -act  $A$ , we obtain the following a representation theorem for the  $\overset{\Delta}{\sim}$ -equivalence classes of the relation  $\overset{\Delta}{\sim}$ , generalizing Theorem 3.2.

**Theorem 3.21.** *Let  $\{G_j\}$  be a family of subgroups  $G_j \subset S$  of a group  $S$ ,  $j \in J$ , only containing exactly one subgroup of each of the conjugate classes of proper non-normal subgroups of a group  $S$ , and  $N_m \triangleleft S$ ,  $m \in M$ , all proper normal subgroups of  $S$ . Then, for any*

$S$ -act  $A \in |_S\mathcal{A}|$ , the  $\triangleq$ -equivalence class  $[A]$  has a representation of exactly one of the following three possible types:

- (i)  $[\coprod_{k \in K} \overline{G_k}^{(*)^2} \coprod_{t \in T} \overline{G_t} \coprod_{l \in L} \overline{N_l}]$ , where  $K, T$ , and  $L$  are some subsets of  $J$  and  $M$ , respectively, and  $K \cap T = \emptyset$ ;
- (ii)  $[\coprod_{k \in K} \overline{G_k} \coprod_{l \in L} \overline{N_l} \coprod z]$ ,  $K \subseteq J, L \subseteq M$ ;
- (iii)  $[\coprod_{k \in K} \overline{G_k} \coprod_{l \in L} \overline{N_l} \coprod z_1 \coprod z_2]$ ,  $K \subseteq J, L \subseteq M$ .

**Proof.** As was mentioned above, from [8, Propositions I.5.17 and I.5.34] one readily gets that any  $S$ -act  $A \in |_S\mathcal{A}|$  is isomorphic to a coproduct of some  $\overline{G_j}$ ,  $j \in J$ , and zero  $S$ -acts. From this, and using a transfinite induction if a group  $S$  contains an infinite number of subgroups, i.e.,  $|J \cup M| \geq \omega$ , the result immediately follows from Propositions 3.14, 3.15, 3.16, 3.17, and Corollary 3.20.  $\square$

As a corollary of Theorem 3.21 and Corollary 3.13, we obtain a simpler version of a representation theorem for the  $\triangleq$ -equivalence classes of  $S$ -acts over abelian groups  $S$ .

**Theorem 3.22.** *Let  $G_j \subset S$ ,  $j \in J$ , be all proper subgroups of an abelian group  $S$ . Then, for any  $S$ -act  $A \in |_S\mathcal{A}|$ , the  $\triangleq$ -equivalence class  $[A]$  has a representation of exactly one of the following three possible types:*

- (i)  $[\coprod_{k \in K} \overline{G_k}]$ ,  $K \subseteq J$ ;
- (ii)  $[\coprod_{k \in K} \overline{G_k} \coprod z]$ ,  $K \subseteq J$ ;
- (iii)  $[\coprod_{k \in K} \overline{G_k} \coprod z_1 \coprod z_2]$ ,  $K \subseteq J$ .  $\square$

**Corollary 3.23.** *In  $_S\mathcal{A}$  over an abelian group  $S$  of a prime order  $p$ , there are only the following three  $\triangleq$ -equivalence classes:  $[S]$ ,  $[S \coprod z]$ , and  $[S \coprod z_1 \coprod z_2]$ .  $\square$*

**Remark 3.24.** Let  $S$  be an abelian group of a prime order  $p$ . Then,  $S \coprod z$  is an injective envelope (see, [8, Section 3.1]) of  $S$ , however,  $S \triangleq S \coprod z$ . Moreover, from Theorem 3.21 it is easy to see that for any  $S$ -act  $A \in |_S\mathcal{A}|$  without zero subacts,  $A$  and its injective envelope are not geometrically equivalent.

Using [8, Theorem 3.1.8 and Exercise 3.1.9], we obtain another corollary of Theorem 3.21.

**Corollary 3.25.** *Let  $G_j \subset S$ ,  $j \in J$ , be all proper subgroups of a group  $S$ . Then, for any injective  $S$ -act  $A \in |_S\mathcal{A}|$ , the  $\triangleq$ -equivalence class  $[A]$  has a representation of exactly one of the following two possible types:*

- (i)  $[\coprod_{k \in K} \overline{G_k} \coprod z]$ ,  $K \subseteq J$ ;
- (ii)  $[\coprod_{k \in K} \overline{G_k} \coprod z_1 \coprod z_2]$ ,  $K \subseteq J$ .  $\square$

Using Proposition 3.16, we also have

**Corollary 3.26.** *In  ${}_S\mathcal{A}$ , all free  $S$ -acts are geometrically equivalent, i.e., they are in  $[S]$ , provided  $|S| > 1$ .  $\square$*

In conclusion, we state the following, in our view, interesting and promising problems.

**Problem 1.** Our conjecture is that in general the  $\triangle$ -equivalence classes  $[A]$  in Theorems 3.21, 3.22, and Corollary 3.25 may have different representations of the same type. Therefore, it would be interesting and important to describe groups  $S$  over which any  $\triangle$ -equivalence class  $[A]$ ,  $A \in {}_S\mathcal{A}$ , has a unique representation of the given types.

**Problem 2.** In light of Corollaries 3.25 and 3.26, in which were considered the classes of injective and free  $S$ -acts, respectively, it is interesting to obtain analogs of those results for other homological classes of  $S$ -acts (see, e.g., [8]).

**Problem 3.** To extend the considerations results of the present paper, obtained for  $S$ -acts over groups  $S$ , to some other interesting varieties  ${}_S\mathcal{A}$  over “nice” monoids  $S$ .

## REFERENCES

- [1] G. Baumslag, A. Myasnikov, and V. N. Remeslennikov, Algebraic geometry over groups, *J. Algebra* **219** (1999) 16–79.
- [2] A. Berzins, B. Plotkin, and E. Plotkin, Algebraic geometry in varieties of algebras with the given algebra of constants, *J. Math. Sci.* **102** (2000) 4039–4070.
- [3] A. Berzins, Geometric equivalence of algebras, *Internat. J. Algebra Comput.* **11** (2001) 447–456.
- [4] Rüdiger Göbel and Saharon Shelah, Radicals and Plotkin’s problem concerning geometrically equivalent groups, *Proc. Amer. Math. Soc.* **130** (2002) 673–674.
- [5] G. Grätzer, *Universal Algebra*, 2nd Ed. (Springer-Verlag, New York-Berlin, 1979).
- [6] Y. Katsov, R. Lipyanski, and B. Plotkin, Automorphisms of categories of free modules, free semimodules, and free Lie modules, *Comm. Algebra* **35** (2007) 931–952.
- [7] Y. Katsov, A problem of B. Plotkin for  $S$ -acts: automorphisms of categories of free  $S$ -acts, *Comm. Algebra* **35** (2007) 1709–1714.
- [8] M. Kilp, U. Knauer, and A. V. Mikhalev, *Monoids, Acts and Categories* (Walter de Gruyter, Berlin-New York, 2000).
- [9] S. Mac Lane, *Categories for the Working Mathematician* (Springer-Verlag, New York-Berlin, 1971).
- [10] A. I. Maltsev, *Algebraic Systems* (Springer-Verlag, New York-Berlin, 1973).
- [11] B. Plotkin, Varieties of algebras and algebraic varieties. Categories of algebraic varieties, *Siberian Adv. Math.* **7** (1997) 64–97.

- [12] B. I. Plotkin, Some notions of algebraic geometry in universal algebra, (Russian) *Algebra i Analiz* **9** (1997) 224–248; English transl., *St. Petersburg Math. J.* **9** (1998) 859–879.
- [13] B. Plotkin, E. Plotkin, A. Tsurkov, Geometrical equivalence of groups, *Comm. Algebra* **27** (1999) 4015–4025.
- [14] B. Plotkin, Seven lectures on the universal algebraic geometry, *Preprint, Institute of Mathematics, Hebrew University, Jerusalem*, arXiv:math. GM/0204245 (2002).
- [15] B. I. Plotkin, Problems in algebra inspired by universal algebraic geometry, (Russian) *Fundam. Prikl. Mat.* **10** (2004) 181–197.
- [16] B. Plotkin, A. Tsurkov, Action type geometrical equivalence of representations of groups, *Preprint* arXiv:math. RT/0501337 (2005).
- [17] A. Tsurkov, Geometrical equivalence of nilpotent torsion free groups, *Preprint* arXiv:math. GR/0411313 (2004).

*Department of Mathematics and Computer Science, Hanover College, Hanover,  
IN 47243-0890, USA*  
*E-mail address: katsou@hanover.edu*